Common Counting Cases

Foundations

1. TOTAL NUMBER OF ELEMENTS IN MULTIPLE DISJOINT SETS

or: number of ways to choose 1 element from the accumulation of multiple disjoint sets

disjoint: separate, no elements in common

|set 1| + |set 2| + ... + |set n|

(rule of sum) ("or")

Example.
There are 80 chips/salsa combinations available at Trader Joe's. There are 21 chips/salsa combinations available at CVS. No brands are shared. Therefore, there are 80+21 = 101 chips/salsa combinations available across both Trader Joe's and CVS.

For the record, I made these numbers up.

2. TOTAL NUMBER OF (SET 1 ELEMENT, ..., SET N ELEMENT) TUPLES

or: number of ways to choose 1 element from set 1, 1 element from set 2, ..., and 1 element from set n

imagine you're filling n slots with an element from each set

_ _ _ _ _ _ _ _ _ _ _ _

where sets are disjoint or where repetition is allowed

|set 1| * |set 2| * ... * |set n|

(rule of product) ("and")

Example.
If I have 5 shirts and 3 skirts, there are 5 * 3 = 15 different shirt/skirt combos I can choose.

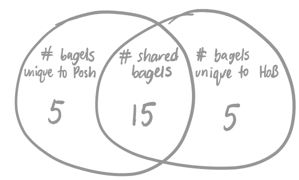
assume each of the shirts and skirts is unique

3. TOTAL NUMBER OF ELEMENTS IN MULTIPLE (MAYBE OVERLAPPING) SETS

$$\sum_{i \in \text{all sets}} |\text{set } i| - \sum_{i,j \in \text{all pairs of sets}} |\text{set } i \cap \text{set } j| + \sum_{i,j,k \in \text{all triplets of sets}} |\text{set } i \cap \text{set } j \cap \text{set } k| - \ldots$$

(inclusion-exclusion principle)
(extension of rule of sum)

Example.
If there are 20 types of bagels at Posh Bagel, 20 types of bagels at House of Bagels, and 15 types of bagels which both stores sell, then in total there are $20+20-15 = 25$ different types of bagels across the two aforementioned bagel shops.



4. TOTAL NUMBER OF ENTITIES WITH AT LEAST ONE SOMETHING

$$\left(\begin{array}{l}\text{total \#} \\ \text{of entities}\end{array}\right) - \left(\begin{array}{l}\text{total \# of entities} \\ \text{with ZERO somethings}\end{array}\right)$$

subtract out the number of elements in the complement of the desired set

alternatively you can count all the cases (one something, two somethings, three somethings, ...) but this is often more difficult

Example.
The number of length-5 ABCD strings with at least one A is $4^5 - 3^5$.

doesn't have to be "at least one"; you can use the complement trick for any set you're trying to count

5. TOTAL NUMBER OF WAYS TO ORDER N ELEMENTS

$$n * (n-1) * \ldots * 1 = n!$$

think of it as filling n slots: n options for the first slot, n−1 options for the second (because no replacement), and so on...

$\underline{n}$ $\underline{n-1}$ $\underline{n-2}$ ... $\underline{1}$

PERMUTATIONS AND COMBINATIONS

TOTAL NUMBER OF WAYS TO CHOOSE K FROM A SELECTION OF N ELEMENTS, WHERE...

6. ORDER MATTERS AND THERE IS REPLACEMENT

n^k

Example.
We can make 5^3 three-letter words from the letters ABCDE.

order
always imagine you're filling k slots.

_ _ _ _ _

now, ask yourself whether

_ _ _ _ _

is equivalent to

_ _ _ _ _

if not, order matters.
if so, order doesn't matter.
or you're colorblind.

7. ORDER MATTERS AND THERE IS NO REPLACEMENT

$$n * (n-1) * \ldots * (n-k+1)$$
$$= \frac{n!}{(n-k)!} = P(n,k)$$

Example.
There are $7 * 6 * 5 * 4$ ways that we can arrange 4 of 7 (different) books into a pile.

8. ORDER DOESN'T MATTER AND THERE IS REPLACEMENT

$$\frac{(k+n-1)!}{(n-1)!\,k!} = \binom{k+n-1}{k}$$

why? basically you're throwing k darts at a space divided by $(n-1)$ hyperplane dividers. then the number of ways we can order k darts and $(n-1)$ dividers is $(k+n-1)!$, divided by $k!$ because the order of the darts doesn't matter and divided by $(n-1)!$ because the order of the dividers doesn't matter.

Example.

If a doughnut store sells 6 types of doughnuts and you want to buy 4, then (assuming they have at least 4 of every doughnut in stock) there are $\binom{9}{4}$ different bags of doughnuts you can end up with.

for a better explanation, see the below SO post (note that they switch k & n):

18

This problem comes by many names - stars and stripes, balls and urns - it's basically a question of how to distribute n objects (call them "balls") into k categories (call them "urns"). We can think of it as follows.

Take n balls and $k-1$ dividers. If a ball falls between two dividers, it goes into the corresponding urn. If there's nothing between two dividers, then there's nothing in the corresponding urn. Let's look at this with a concrete example.

I want to distribute 5 balls into 3 urns. As before, take 5 balls and 2 dividers. Visually:

|ooo|oo

In this order, we'd have nothing in the first urn, three in the second urn and two balls in the third urn. The question then is how many ways can we arrange these 5 balls and two dividers? Clearly: $\frac{(5+3-1)!}{5!(3-1)!} = \binom{7}{2} = \binom{7}{5}$.

We have that there are $\frac{(n+(k-1))!}{(k-1)!n!}$ the n balls and $k-1$ dividers (since the balls aren't distinct from each other and the dividers aren't distinct from each other). Notice that this is equal to $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$.

9. ORDER DOESN'T MATTER AND THERE IS NO REPLACEMENT

$$\frac{n*(n-1)*\ldots*(n-k+1)}{k!}$$

$$= \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

$k!$ is the number of different orderings of k elements. if order doesn't matter, all of these permutations should be treated as the same. therefore, we have to additionally divide out by $k!$ to avoid counting each group of k elements as $k!$ distinct permutations.

(${}_nP_k$ counts each combination $k!$ times; it treats _•_ _•_ _•_ and _•_ _•_ _•_ as being different, since the ordering is different.)