Epipolar Constraint Derivation

For the Fundamental Matrix

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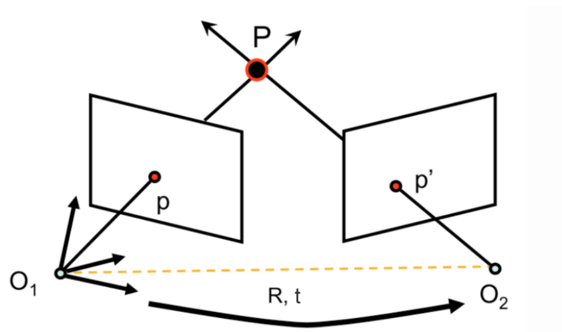

Let p, p' be corresponding 2D points in image 1 and 2 respectively, with cameras related by rotation R and translation t.[1] Let the intrinsic matrices of cameras 1 and 2 be K and K' respectively.

Then p' in camera 1 coordinates[2] is

$$R^T((K')^{-1}p' - t) = R^T(K')^{-1}p' - R^T t$$

This point, along with $R^T t$, lies in the epipolar plane.[3] So a normal to the epipolar plane is

$$\begin{aligned} R^T t \times (R^T(K')^{-1}p' - R^T t) &= R^T t \times R^T(K')^{-1}p' - R^T t \times R^T t \\ &= R^T(t \times (K')^{-1}p') \end{aligned}$$

Since $K^{-1}p$ also lies in the epipolar plane, this normal dotted with $K^{-1}p$ should equal 0:

$$\begin{aligned} (R^T(t \times (K')^{-1}p'))^T K^{-1}p &= 0 \\ (t \times (K')^{-1}p')^T R K^{-1}p &= 0 \\ (T_x(K')^{-1}p')^T R K^{-1}p &= 0 \\ p'^T(K')^{-T}T_x^T R K^{-1}p &= 0 \\ p'^T(K')^{-T}T_x R K^{-1}p &= 0 \\ p'^T F p &= 0 \end{aligned}$$

where $F = (K')^{-T}T_x R K^{-1}$ is the fundamental matrix.
T_x is the matrix form of the cross product and is constructed from the translation vector t:

$$\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$$

[1] Specifically, R, t is the transformation from camera 2's frame to camera 1's frame, meaning the relationship between a location in camera 1 coordinates (p_1) and the same location in camera 2 coordinates (p_2) is $p_2 = Rp_1 + t$.

[2] We define "camera coords" as points in the camera frame, and "image coords" as projected points in the image.

[3] Under our formulation, $-R^T t$ is the vector in the camera 1 frame from O_1 to O_2 ($\mathbf{0}$ in camera 2 coordinates).