

CS 61A: Orders of Growth

6 December 2016

Orders of Growth: Boring Text Version

An order of growth is a **function** describing how something scales (usually a resource; time or space) with respect to some input.

You'll hear things like this: “what is the *order of growth* of <function-name>'s running time, in terms of <input-name(s)>?”

Common orders of growth:

$O(1)$, $O(\log n)$, $O(n)$, $O(n \log n)$, $O(n^c)$, $O(c^n)$... [pretty much any mathematical function of the input can be an order of growth]

Orders of Growth: More Boring Text

Orders of growth allow us to assert that certain functions run more quickly (as a function of their input), or **scale better**, than others.

This is what happens when all of the 170 students run their ~~garbage~~ $O(n^4)$ -time project code on the instructional machines¹ →

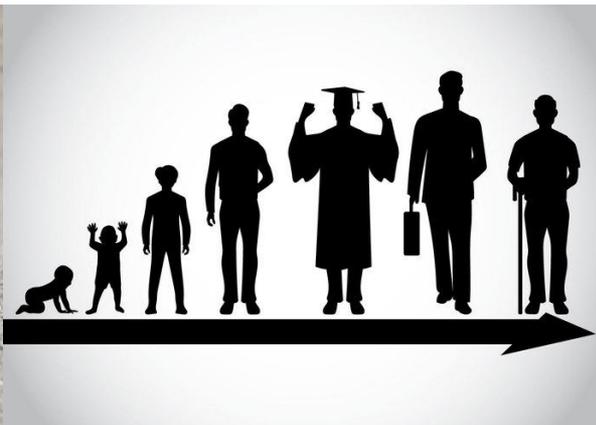
This site displays usage stats for the Berkeley EECS instructional computers.
The data below was gathered **a few seconds ago**. Refresh to check for updates.

The least-busy Hive server is hive17.cs.berkeley.edu.

| Server name ↕ | Overall load ² ↕ | User count ↕ | CPU usage ² ↓ |
|------------------------|-----------------------------|--------------|--------------------------|
| hive19 | ● HIGH | 8 users | 11076.13% |
| hive21 | ● HIGH | 11 users | 6857.75% |
| hive20 | ● HIGH | 9 users | 6010.38% |
| hive15 | ● HIGH | 12 users | 5967.38% |
| hive7 | ● HIGH | 6 users | 5697.88% |
| hive5 | ● HIGH | 18 users | 5680.63% |
| hive8 | ● HIGH | 11 users | 5678.63% |
| hive3 | ● HIGH | 15 users | 5677.13% |
| hive10 | ● HIGH | 16 users | 5640.13% |

¹ This isn't actually that great a representation of runtime growth, since a bunch of people are probably just looping indefinitely and running lots of instances.

Orders of Growth: (Also Boring) Visual Examples



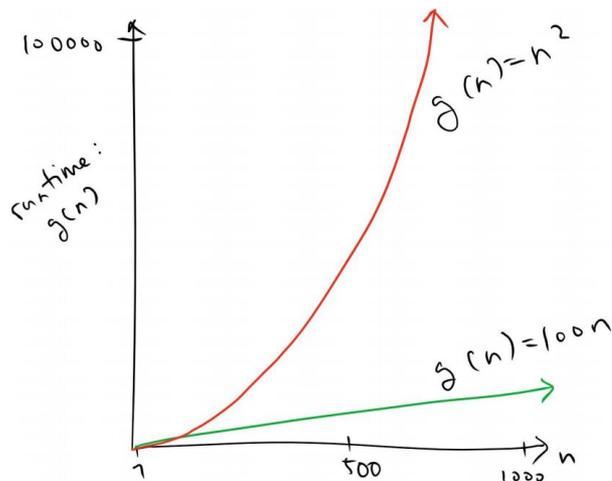
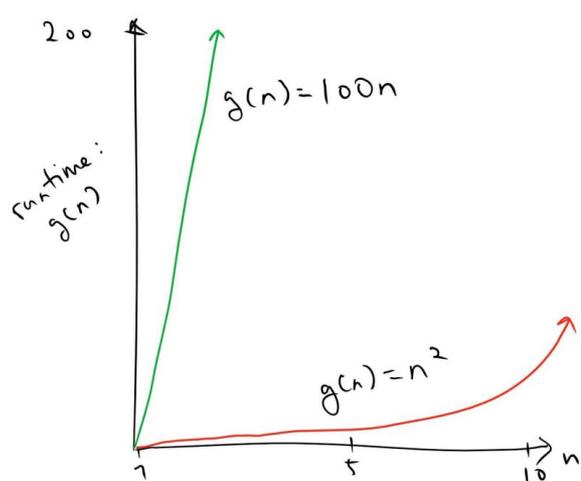
A rock; we could argue that its growth is an $O(1)$ function of time

Age increases linearly over time, of course

The wheat and chessboard problem

A Rough Approximation

We don't care about small inputs; we can always handle those pretty easily anyway. We care about what happens as the input size gets REALLY BIG. Small input sizes aren't necessarily representative anyway:



(Thanks to Jerome Baek for this great visual!)

A Rough Approximation, continued

Asymptotically, only the highest-order term (or terms if there are multiple input variables) in the growth function matters. Therefore, that's the only term we retain.

ex. $n^3 + 40000n^{2.1} + 26$ becomes n^3

For similar reasons we'll also omit constant multipliers for that first term (it helps with standardization, and anyway we want to stay focused on the big - asymptotic - picture).

$65\sqrt{n} + \log n + \log(\log n)$ becomes \sqrt{n}

Summary So Far

An order of growth is just a function that depicts how stuff (like running time) scales. $O(f(n))$ means that aforementioned “stuff” increases no faster than the $f(n)$ -class of functions as n gets larger and larger. This can be useful for guaranteeing efficiency in the temporal or spatial domain.

When determining an order of growth, do two things:

1. Drop lower-order terms.
2. Drop multiplicative constants.

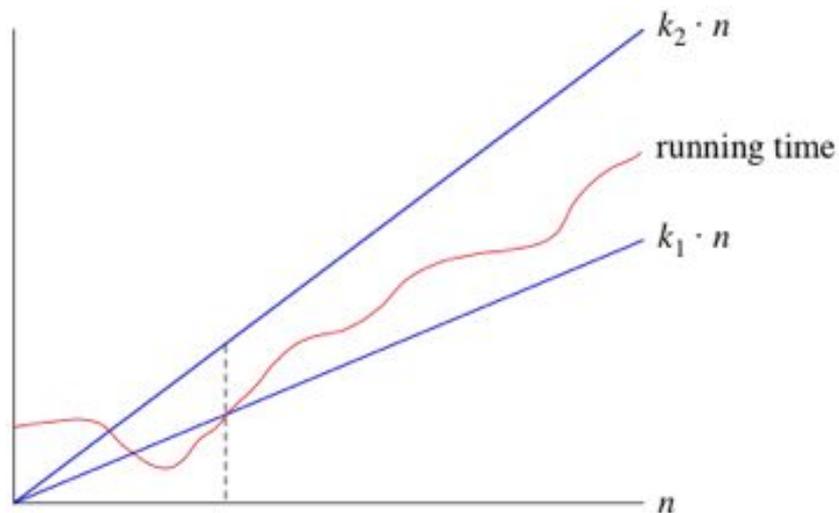
Neither descriptor is asymptotically relevant.

Vaguely Mathematical Depiction (Out of Scope for 61A)

Big-Theta

The definition of Big-Theta:

If we say that the order of growth is $f(n)$, then there exist positive constants k_1 and k_2 for which the ACTUAL order of growth is sandwiched between $k_1 \cdot f(n)$ and $k_2 \cdot f(n)$... for sufficiently large values of n .

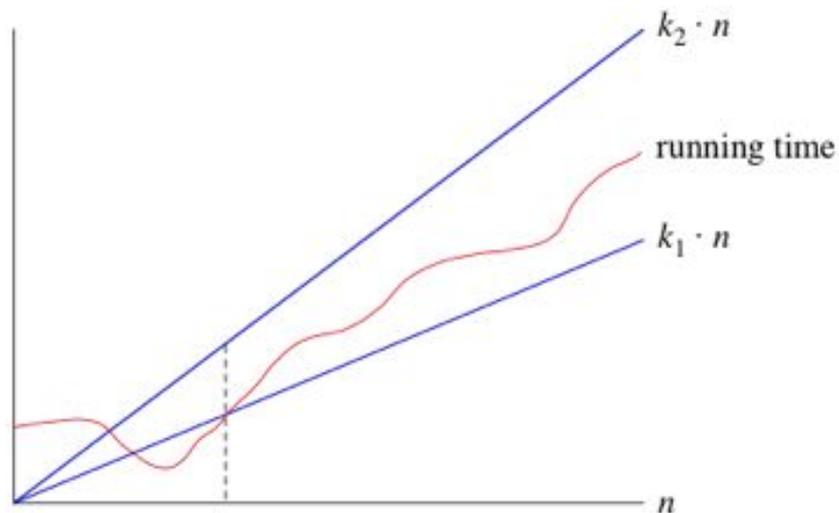


In this diagram, the “actual” order of growth is the red line. The blue lines are the upper and lower bounds.

Big-Theta, cont.

If the growth function (runtime as a function of input size) is ALWAYS sandwiched between $0.5n$ and $1n$ when n is really large, then the order of growth would be $\Theta(n)$.

In other words, if $k_1 = 0.5$ and $k_2 = 1$ in the diagram to the right, then the running time can be said to be $\Theta(n)$.

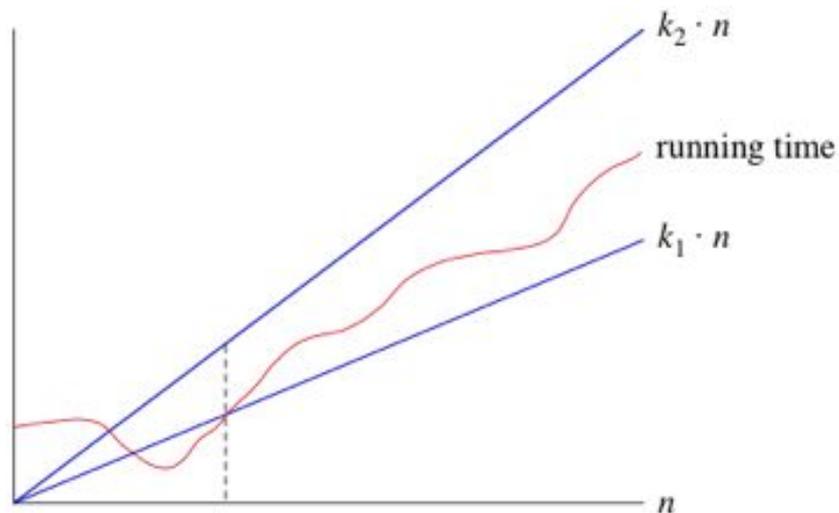


Big-O

The definition of Big-O:

If we say that the order of growth is $f(n)$, then there exists a constant k_2 for which the ACTUAL order of growth is BELOW $k_2 \cdot f(n)$ for sufficiently large values of n .

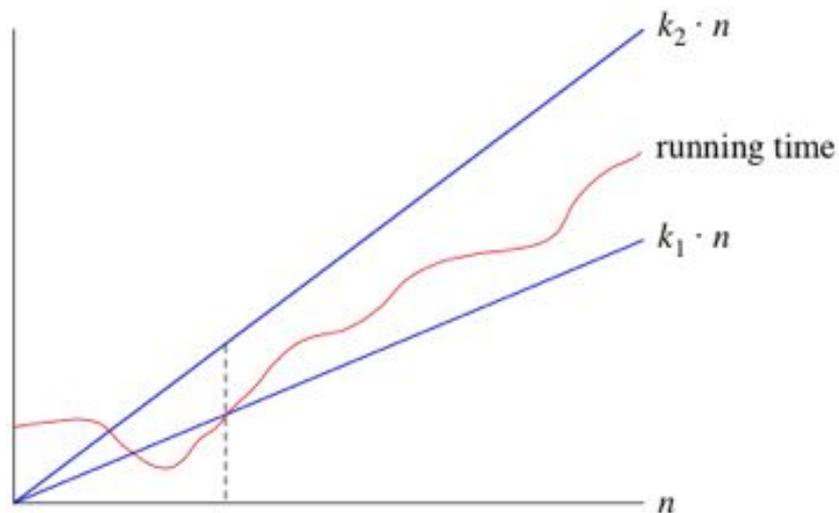
(Basically, it's only the upper bound.)



Big-O vs Big-Theta

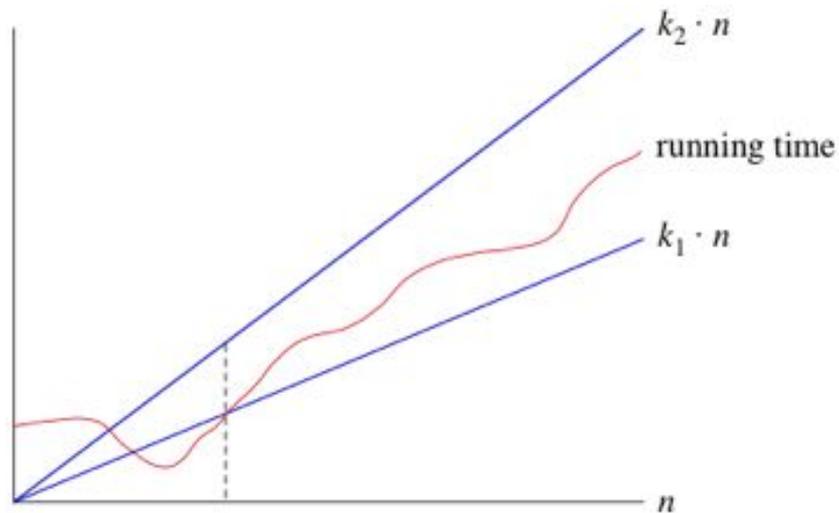
In this class (and in practice), you try to use Big-O as Big-Theta because it's not very informative otherwise (i.e. pretty much everything is *technically* $O(n^n)$, but who cares?).

tl;dr Even though Big-O technically only refers to an upper bound, we want you to find the “tightest” bound.



Big-O vs Big-Theta

So if $k_1 = 0.5$ and $k_2 = 1$ in our diagram, we'd say that the order of growth is $\Theta(n) \approx O(n)$.



Determining Order of Growth

Approach

If faced with a function of unknown time complexity:

Go through the function line-by-line, determining approximately how much time each block of code takes as a function of n . Then sum them all together (by “them” I mean your estimation for each region of code) and drop lower-order terms. That’s pretty much it.

If there’s recursion (which there will be), figure out how much work there is to be done in each call and how many calls there’ll be. Then multiply those things together.

An Example [Summer 2012 Final | Q2(c)]

```
def carpe_noctem(n):  
    if n <= 1:  
        return n  
    return carpe_noctem(n - 1) \  
        + carpe_noctem(n - 2)
```

```
def yolo(n):  
    if n <= 1:  
        return 5  
    sum = 0  
    for i in range(n):  
        sum += carpe_noctem(n)  
    return sum + yolo(n - 1)
```

Question: What is the order of growth in n of the runtime of `yolo`, where n is its input?

Answer:

- Well, going through each line...

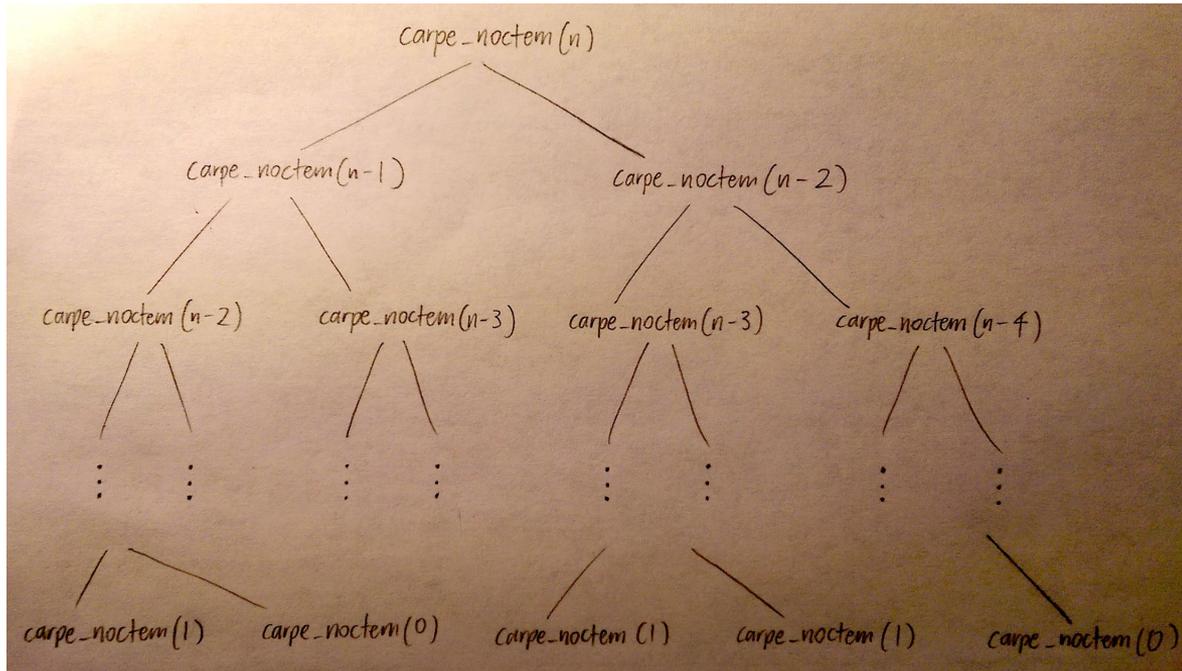
An Example [Summer 2012 Final | Q2(c)]

```
def carpe_noctem(n):
    if n <= 1: # this block is O(1) on its own
        return n
    return carpe_noctem(n - 1) + carpe_noctem(n - 2) # TWO recursive calls

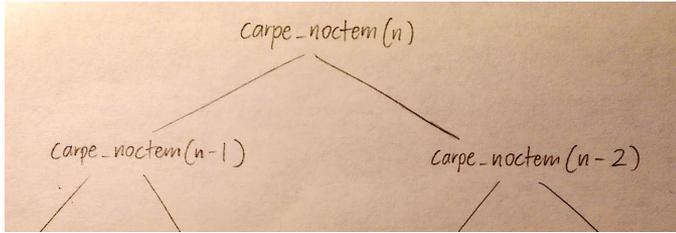
def yolo(n):
    if n <= 1: # this block is O(1) on its own
        return 5
    sum = 0 # O(1)
    for i in range(n): # this block is O(n) times...
        sum += carpe_noctem(n) # whatever the OOG of carpe_noctem is
    return sum + yolo(n - 1) # and then there's another recursive call
```

An Example [Summer 2012 Final | Q2(c)]

Recursive call tree for `carpe_noctem` (authentic handwritten edition):



An Example [Summer 2012 Final | Q2(c)]



The call tree is a binary tree of depth $(n - 1)$. There are at most $2^{k+1} - 1$ nodes in a binary tree of depth k , which means that there are at most $2^{(n-1)+1} - 1 = 2^n - 1$ nodes in *this* tree.

Each of these nodes represents a call to `carpe_noctem`, and we do $O(1)$ work in the body of each of these calls, so a single `carpe_noctem` call produces $O(2^n)$ recursive `carpe_noctem` calls and $O(1) * O(2^n) = O(2^n)$ work overall. **Conclusion:** `carpe_noctem`'s growth function is $O(2^n)$.

An Example [Summer 2012 Final | Q2(c)]

$yolo(n) \rightarrow yolo(n-1) \rightarrow yolo(n-2) \rightarrow \dots \rightarrow yolo(2) \rightarrow yolo(1)$

The call sequence for `yolo`, meanwhile, is a lot simpler. It should be clear from the above sketch that `yolo(n)` involves `n` calls to `yolo`. Accordingly, we know that the stuff in the body of `yolo` happens `n` times in total.

An Example [Summer 2012 Final | Q2(c)]

```
def carpe_noctem(n):  
    # stuff happens here, but all we need to know is that it's  $O(2^n)$   
  
def yolo(n):  
    #  $O(1)$  stuff  
    for i in range(n): okay, we do the stuff in the loop  $n$  times  
        sum += carpe_noctem(n) # carpe_noctem is  $O(2^n)$   
    return sum + yolo(n - 1) # and then all of this stuff happens  $n$  times again
```

In conclusion: the body of yolo is $O(n * 2^n)$, and then that code gets executed n times. Therefore yolo is, holistically speaking, an $O(n * n * 2^n) = O(n^2 * 2^n)$ function because that's how much work we do as a result of one yolo(n) call.

General Heuristics (Not Guarantees!)

Warning: this stuff definitely isn't always going to be true (especially in the case of exam questions designed to filter out confused students).

- When there are c recursive calls in the function body (tree recursion), there tends to be $O(c^n)$ calls overall (exponential growth).
- Double-nested for-loops tend to indicate that you'll do the stuff in the inner loop n^2 times.
- If you make multiplicative progress during every step (e.g. by dividing problem size by 2 or multiplying something by 3), it's likely logarithmic growth.

Individual Function Descriptions (+ Exercises)

Live Answer Submission Link

(Process: I review one classification at a time, and present 2-3 functions after each one. Your job is to identify the order of growth of each function's running time, where the answer will tend to focus on the classifications I've already been through. There may or may not be exceptions, though.)

It'd be great if you guys can anonymously submit your answers as we go, so I can see how you're doing and figure out how difficult everything is.

Link to live / anonymous poll: **[edit: done through Poll Everywhere, see alt. version of slides]**

DISCLAIMER In the following slides, we treat the growth function as a runtime descriptor. However, note that orders of growth can be used to describe *any* phenomena that scale as a function of their inputs (memory is another big one, for example).



$O(1)$

Constant time. Best order of growth for scalability; runtime is not affected by the input size.

```
def const(n):  
    n = 902 + 54  
    return 'hamburger'
```

$O(\log n)$

Logarithmic time. Amazingly scalable; a multiplicative increase in input size leads to an additive increase in running time.

```
def loga(n):  
    if n <= 1:  
        return 1  
    return n * loga(n // 2)
```

Exercise 1

```
def mystery1(n):
    n, result = str(n), ''
    num_digits = len(n)
    for i in range(num_digits):
        result += n[num_digits - i - 1]
    return result
```

```
def mystery2(n):
    n, result = 5, 0
    while n <= 3000:
        result += mystery1(n // 2)
        n += 1
    return result
```

Reminder: we want the order of growth of the runtime a function of n .

Example answers: $O(1)$, $O(n)$, $O(n^2)$...

Exercise 1 Solutions

```
def mystery1(n):
    n, result = str(n), ''
    num_digits = len(n)
    for i in range(num_digits):
        result += n[num_digits - i - 1]
    return result
```

$O(\log n)$.

Notes: `str(n)` is $O(\log n)$, `len(n)` is $O(1)$.

```
def mystery2(n):
    n, result = 5, 0
    while n <= 3000:
        result += mystery1(n // 2)
        n += 1
    return result
```

$O(1)$.

Notes: The input `n` doesn't even matter!

Exercise 1 Solutions

```
def mystery1(n):
    n, result = str(n), ''
    num_digits = len(n)
    for i in range(num_digits):
        result += n[num_digits - i - 1]
    return result
```

$O(\log n)$.

I anticipate this one being confusing (although I would love for you guys to prove me wrong so I don't have to use this slide!). Let's go through it a bit.

1. $\text{str}(n)$ is $O(\log n)$ because you have to MULTIPLY n by your radix in order to ADD one digit to your output string.
2. $\text{len}(n)$ is $O(1)$ because Python strings keep track of their own length, but you should realize that it's at worst $O(\log n)$ for the same reasons as 1 (there are only $O(\log n)$ digits).
3. Since there are only $O(\log n)$ digits, the loop simply performs constant-time¹ indexing and addition $O(\log n)$ times.

¹ We're not interested in bit-level complexity here.

$O(\sqrt{n})$

Square-root time, aka knockoff logarithmic time (runtime still increases slowly with input size). Better than $O(n)$, but rarely observed.

```
def sqroot(n):  
    lim = int(sqrt(n))  
    for i in range(lim):  
        n += 45  
    return n
```

Exercise 2

```
def mystery3(n):  
    if n < 0 or n <= sqrt(n):  
        return n  
    return n + mystery3(n // 3)
```

```
def mystery4(n):  
    if n < 0 or sqrt(n) <= 50:  
        return 1  
    return n * mystery4(n // 2)
```

```
def mystery5(n):  
    for _ in range(int(sqrt(n))):  
        n = 1 + 1  
    return n
```

Exercise 2 Solutions

```
def mystery3(n):  
    if n < 0 or n <= sqrt(n):  
        return n  
    return n + mystery3(n // 3)
```

$O(\log n)$. $n \leq \sqrt{n}$ will only be hit when $0 \leq n \leq 1$.

```
def mystery5(n):  
    for _ in range(int(sqrt(n))):  
        n = 1 + 1  
    return n
```

$O(\sqrt{n})$. Possible confusion: \sqrt{n} is only computed once at the beginning.

```
def mystery4(n):  
    if n < 0 or sqrt(n) <= 50:  
        return 1  
    return n * mystery4(n // 2)
```

$O(\log n)$. $\sqrt{n} \leq 50$ is equivalent to $n \leq 2500$, so this ends up being a standard logarithmic-time algorithm.

$O(n)$

Linear time. Still very scalable; adding a constant to the input size also adds a constant to the runtime.

```
def lin(n):  
    if n <= 1:  
        return 1  
    return n + lin(n - 1)
```

Exercise 3

— — —

```
def mystery6(n):
    while n > 1:
        x = n
        while x > 1:
            print(n, x)
            x = x // 2
        n -= 1
```

(Thanks to Mark Miyashita for this problem!)

```
def mystery7(n):
    result = 0
    for i in range(n // 10):
        result += 1
    for j in range(10):
        result += 1
    for k in range(10 // n):
        result += 1
    return result
```

Exercise 3 Solutions

```
def mystery6(n):
    while n > 1:
        x = n
        while x > 1:
            print(n, x)
            x = x // 2
        n -= 1
```

$O(n \log n)$. Inner loop is $O(\log n)$, and it happens $O(n)$ times.

```
def mystery7(n):
    result = 0
    for i in range(n // 10):
        result += 1
        for j in range(10):
            result += 1
            for k in range(10 // n):
                result += 1
    return result
```

$O(n)$. The number of iterations in the j-loop is based on a constant, and the k-loop won't happen once n is sufficiently large.

$O(n^2)$

Quadratic time. Still polynomial, so it could be worse; multiplying input size by a constant factor ends up multiplying the runtime by the square of that factor.

```
def quad(n):  
    if n <= 1:  
        return 1  
    r = lin(n) * quad(n - 1)  
    return r
```

Exercise 4

```
def mystery8(n):
    if n == 0: return ''
    result, stringified = '', str(n)
    for digit in stringified:
        for _ in range(n):
            result += digit
    result += mystery8(n - 1)
    return result
```

```
def mystery9(n):
    total = 0
    for i in range(1, n):
        total *= 2
        if i % n == 0:
            total *= mystery9(n - 1)
            total *= mystery9(n - 2)
        elif i == n // 2:
            for j in range(1, n):
                total *= j
    return total
```

Exercise 4 Solutions

```
def mystery8(n):
    if n == 0: return ''
    result, stringified = '', str(n)
    for digit in stringified:
        for _ in range(n):
            result += digit
    result += mystery8(n - 1)
    return result
```

$O(n^2 \log n)$. The double-nested loop is $n \log n$ work. And we run it n times (because of the recursive `mystery8` call).

```
def mystery9(n):
    total = 0
    for i in range(1, n):
        total *= 2
        if i % n == 0:
            total *= mystery9(n - 1)
            total *= mystery9(n - 2)
        elif i == n // 2:
            for j in range(1, n):
                total *= j
    return total
```

$O(n)$. The first `if`-statement never happens, and the second only happens once.

$O(2^n)$

Exponential time. Not scalable at all; identifies problems as intractable. Adding to the input size multiplies the runtime.

```
def expo(n):  
    if n <= 1:  
        return 1  
    r1 = expo(n - 1) + 1  
    r2 = expo(n - 1) + 2  
    return r1 * r2
```

A General Timing Comparison

| | n = 10 | n = 50 | n = 100 | n = 1000 |
|----------------|---------|----------------------|-----------------------|------------------------|
| logn | 0.0003s | 0.0006s | 0.0007s | 0.0010s |
| sqrt(n) | 0.0003s | 0.0007s | 0.0010s | 0.0032s |
| n | 0.0010s | 0.0050s | 0.0100s | 0.1000s |
| nlogn | 0.0033s | 0.0282s | 0.0664s | 0.9966s |
| n ² | 0.0100s | 0.2500s | 1.0000s | 100.00s |
| n ⁶ | 1.6667m | 18.102d | 3.1710y | 3171.0c |
| 2 ⁿ | 0.1024s | 35.702c | 4x10 ¹⁶ c | 1x10 ¹⁶⁶ c |
| n! | 362.88s | 1x10 ⁵¹ c | 3x10 ¹⁴⁴ c | 1x10 ²⁵⁵⁴ c |

← Time required to process n items at a speed of 10,000 operations per second, using eight different algorithms

s = seconds
m = minutes
d = days
y = years
c = *centuries*

Source:
<http://www.ccs.neu.edu/home/jaa/CS7800.12F/Information/Handouts/order.html>

Exercise 5

— — —

```
def mystery10(n):  
    if n > 0:  
        r1 = mystery10(-n)  
        r2 = mystery10(n - 1)  
        return r1 + r2  
    return 1
```

```
def mystery11(n):  
    if n < 1: return n  
    def mystery12(n):  
        i = 1  
        while i < n:  
            i *= 2  
        return i  
    return mystery11(n / 2) + mystery11(n / 2) \  
        + mystery12(n - 2)
```

Exercise 5 Solutions

```
def mystery10(n):
    if n > 0:
        r1 = mystery10(-n)
        r2 = mystery10(n - 1)
        return r1 + r2
    return 1
```

$O(n)$. The first recursive call can never go anywhere.

```
def mystery11(n):
    if n < 1: return n
    def mystery12(n):
        i = 1
        while i < n:
            i *= 2
        return i
    return mystery11(n / 2) + mystery11(n / 2) \
        + mystery12(n - 2)
```

$O(n \log n)$. We make $O(2^{\log n}) = O(n)$ recursive calls, and each recursive call does $\log n$ work.

A Few More Exercises

Exercise 6

The orders of growth should now be functions of m **and** n .

```
def mystery13(m, n):
    if n <= 1:
        return 0
    result = 0
    for i in range(3 ** m):
        result += i // n
    return result + mystery13(m - 5, n // 3)
```

```
def mystery14(m, n):
    result = 0
    for i in range(1, m):
        j = i * i
        while j <= n:
            result, j = result + j, j + 1
    return result
```

Exercise 6 Solutions

The orders of growth should now be functions of m **and** n .

```
def mystery13(m, n):
    if n <= 1:
        return 0
    result = 0
    for i in range(3 ** m):
        result += i // n
    return result + mystery13(m - 5, n // 3)
```

$O(3^m \log n)$. Work done in body is $O(3^m)$, with $O(\log_3 n)$ calls to the function (bases for logs don't matter because a change of base is just a constant multiplication).

```
def mystery14(m, n):
    result = 0
    for i in range(1, m):
        j = i * i
        while j <= n:
            result, j = result + j, j + 1
    return result
```

$O(m + n\sqrt{n})$. The outer loop happens m times no matter what (doing guaranteed constant work), while the inner loop only runs when $i \leq \sqrt{n}$ (i.e. it does n work \sqrt{n} times).

Exercise 7

— — —

Define n to be the length of the input list. How much memory does the following program use as a function of n ?

```
def weighted_random_choice(lst):  
    temp = []  
    for i in range(len(lst)):  
        temp.extend([lst[i]] * (i + 1))  
    return random.choice(temp)
```

Exercise 7 Solutions

Define n to be the length of the input list. How much memory does the following program use as a function of n ?

```
def weighted_random_choice(lst):  
    temp = []  
    for i in range(len(lst)):  
        temp.extend([lst[i]] * (i + 1))  
    return random.choice(temp)
```

$O(n^2)$. The length of the temporary list is $1 + 2 + 3 + \dots + n$, which we know (through Gauss's summing trick) is equal to $n(n + 1) / 2 = O(n^2)$.

Exercise 8

— — —

Provide an algorithm that, given a sorted list A of distinct integers, determines whether there is an index i for which $A[i] = i$. Your algorithm should run in time $O(\log n)$, where n is the length of the list. *You don't have to write actual code for this; pseudocode or a general approach would be sufficient.*

Thanks to CS 170 for the question description!
If you want to write code, there's a skeleton to the right.

```
def index_exists(A):  
    def helper(lower, upper):  
        if _____:  
            return _____  
        mid_idx = (lower + upper) // 2  
        if _____:  
            return True  
        elif _____:  
            return _____  
        else:  
            return _____  
    return _____
```

Exercise 8 Solutions

Provide an algorithm that, given a sorted list A of distinct integers, determines whether there is an index i for which $A[i] = i$. Your algorithm should run in time $O(\log n)$, where n is the length of the list. *You don't have to write actual code for this; pseudocode or a general approach would be sufficient.*

Thanks to CS 170 for the question description!
If you want to write code, there's a skeleton to the right.

```
def index_exists(A):
    def helper(lower, upper):
        if lower >= upper:
            return A[upper] == upper
        mid_idx = (lower + upper) // 2
        if A[mid_idx] == mid_idx:
            return True
        elif A[mid_idx] > mid_idx:
            return helper(lower, mid_idx - 1)
        else:
            return helper(mid_idx + 1, upper)
    return helper(0, len(A) - 1)
```

NOTE I intended for these problems to be very tricky. (At the time of me writing this, I'm not sure whether I succeeded.) That being said, some of these questions are probably slightly above the level of difficulty you'd expect on the final. Even if you didn't get a lot of them, you might not have to worry TOO much.



Past Exam Questions

Summer 2013 MT2 | Q2(a)

```
def fizzle(n):  
    if n <= 0:  
        return n  
    elif n % 23 == 0:  
        return n  
    return fizzle(n - 1)
```

What is the order of growth for a call to fizzle(n)?

Summer 2013 MT2 | Q2(a) Solutions

```
def fizzle(n):  
    if n <= 0:  
        return n  
    elif n % 23 == 0: # this line ensures that fizzle will never be called more than 23 times  
        return n  
    return fizzle(n - 1)
```

What is the order of growth for a call to fizzle(n)?

Answer: $O(1)$.

Summer 2013 MT2 | Q2(b)

— — —

```
def boom(n):  
    if n == 0: return 'BOOM!'  
    return boom(n - 1)
```

```
def explode(n):  
    if n == 0: return boom(n)  
    i = 0  
    while i < n:  
        boom(n)  
        i += 1  
    return boom(n)
```

What is the order of growth for a call to `explode(n)`?

Summer 2013 MT2 | Q2(b) Solutions

```
def boom(n):
    if n == 0: return 'BOOM!'
    return boom(n - 1)

def explode(n):
    if n == 0: return boom(n)
    i = 0
    while i < n:
        boom(n) # n work (happening n times because of the loop)
        i += 1
    return boom(n)
```

What is the order of growth for a call to `explode(n)`? $O(n^2)$.

Summer 2013 MT2 | Q2(c)

```
def dreams(n):  
    if n <= 0:  
        return n  
    if n > 0:  
        return n + dreams(n // 2)
```

What is the order of growth for a call to `dreams(n)`?

Summer 2013 MT2 | Q2(c) Solutions

```
def dreams(n):  
    if n <= 0:  
        return n  
    if n > 0:  
        return n + dreams(n // 2) # divide the problem in half every time
```

What is the order of growth for a call to `dreams(n)`?

Answer: $O(\log n)$.

Spring 2014 MT2 | Q6(a)

Consider the following function (assume that parameter S is a list):

```
def umatches(S):  
    result = set()  
    for item in S:  
        if item in result:  
            result.remove(item)  
        else:  
            result.add(item)  
    return result
```

Fill in the blank: The function umatches returns the set of all

-----.

Spring 2014 MT2 | Q6(a) Solutions

Consider the following function (assume that parameter S is a list):

```
def umatches(S):
    result = set()
    for item in S:
        if item in result:
            result.remove(item)
        else:
            result.add(item)
    return result
```

Fill in the blank: The function `umatches` returns the set of all values in S that occur an odd number of times.

Spring 2014 MT2 | Q6(b)

```
def umatches(S):
    result = set()
    for item in S:
        if item in result:
            result.remove(item)
        else:
            result.add(item)
    return result
```

Let's assume that the operations of adding to, removing from, or checking containment in a set each take roughly constant time. Give an asymptotic bound (the tightest you can) on the worst-case time for `umatches` as a function of $N = \text{len}(S)$.

Spring 2014 MT2 | Q6(b) Solutions

```
def umatches(S):
    result = set()
    for item in S: # this is why it's O(N)
        if item in result:
            result.remove(item)
        else:
            result.add(item)
    return result
```

Let's assume that the operations of adding to, removing from, or checking containment in a set each take roughly constant time. Give an asymptotic bound (the tightest you can) on the worst-case time for `umatches` as a function of $N = \text{len}(S)$.

Answer: $O(N)$.

Spring 2014 MT2 | Q6(c)

```
def umatches(S):
    result = []
    for item in S:
        if item in result:
            result.remove(item)
        else:
            result.append(item)
    return result
```

Suppose that instead of having result be a set, we make it a list (so that it is initialized to [] and we use .append to add an item; **changes shown to the left**). What now is the worst-case time bound? You can assume that .append is a constant-time operation, and .remove and the in operator require time that is $\Theta(L)$ in the worst case, where L is the length of the list operated on. Since we never add an item that is already in the list, each value appears at most once, just as for a Python set.

Spring 2014 MT2 | Q6(c) Solutions

```
def umatches(S):
    result = []
    for item in S:
        if item in result:
            result.remove(item)
        else:
            result.append(item)
    return result
```

Suppose that instead of having result be a set, we make it a list (so that it is initialized to [] and we use .append to add an item; **changes shown to the left**). What now is the worst-case time bound? You can assume that .append is a constant-time operation, and .remove and the in operator require time that is $\Theta(L)$ in the worst case, where L is the length of the list operated on. Since we never add an item that is already in the list, each value appears at most once, just as for a Python set.

Answer: $O(N^2)$. In the worst case, where every item in S is the same, you have to do two $\Theta(L)$ operations (in and .remove) for $N / 2$ items in S . Since L is really $O(N)$, we have an $O(N^2)$ function overall.

Spring 2014 MT2 | Q6(d)

```
def umatches(S):
    result = []
    for item in S:
        if item in result:
            result.remove(item)
        else:
            result.append(item)
    return result
```

Now suppose that we consider only cases where the number of different values in list S is at most 100, and we again use a list for `result`. What is the worst-case time now?

Spring 2014 MT2 | Q6(d) Solutions

```
def umatches(S):
    result = []
    for item in S:
        if item in result:
            result.remove(item)
        else:
            result.append(item)
    return result
```

Now suppose that we consider only cases where the number of different values in list S is at most 100, and we again use a list for $result$. What is the worst-case time now?

Answer: $O(N)$. L is now upper bounded by 100, so $\Theta(L)$ becomes $\Theta(1)$.

Summer 2015 MT2 | Q5(d)

```
def append(link, value):
    """Mutates LINK by adding VALUE to
    the end of LINK.
    """
    if link.rest is Link.empty:
        link.rest = Link(value)
    else:
        append(link.rest, value)
```

```
def extend(link1, link2):
    """Mutates LINK_1 so that all
    elements of LINK_2 are added to the
    end of LINK_1.
    """
    while link2 is not Link.empty:
        append(link1, link2.first)
        link2 = link2.rest
```

(i) What order of growth describes the runtime of calling `append`? Give your function in terms of n , where n is the number of elements in the input `LINK`.

(ii) Assuming the two input linked lists both contain n elements, what order of growth best describes the runtime of calling `extend`?

Summer 2015 MT2 | Q5(d) Solutions

```
def append(link, value):
    """Mutates LINK by adding VALUE to
    the end of LINK.
    """
    if link.rest is Link.empty:
        link.rest = Link(value)
    else:
        append(link.rest, value)
```

```
def extend(link1, link2):
    """Mutates LINK_1 so that all
    elements of LINK_2 are added to the
    end of LINK_1.
    """
    while link2 is not Link.empty:
        append(link1, link2.first)
        link2 = link2.rest
```

(i) What order of growth describes the runtime of calling `append`? Give your function in terms of n , where n is the number of elements in the input `LINK`. **Answer: $O(n)$.**

(ii) Assuming the two input linked lists both contain n elements, what order of growth best describes the runtime of calling `extend`? **Answer: $O(n^2)$.**

Summer 2012 Final | Q2(a)

```
def collide(n):
    lst = []
    for i in range(n):
        lst.append(i)
    if n <= 1:
        return 1
    if n <= 50:
        return collide(n - 1) + collide(n - 2)
    elif n > 50:
        return collide(50) + collide(49)
```

What is the order of growth in n of the runtime of `collide`, where n is its input?

Summer 2012 Final | Q2(a) Solutions

```
def collide(n):
    lst = []
    for i in range(n): # 0(n) block of code right here
        lst.append(i)
    if n <= 1:
        return 1
    if n <= 50:
        return collide(n - 1) + collide(n - 2)
    elif n > 50: # this covers the case we're interested in (really large n)
        return collide(50) + collide(49)
```

What is the order of growth in n of the runtime of `collide`, where n is its input?

Answer: $O(n)$. For large n , it performs an $O(n)$ list initialization and then runs `collide(50) + collide(49)`. Since 50 and 49 are constants, that part's runtime is irrespective of n .

Summer 2012 Final | Q2(b)

```
def crash(n):  
    if n < 1:  
        return n  
    return crash(n - 1) * n
```

What is the order of growth in n of the runtime of `into_me`, where n is its input?

```
def into_me(n):  
    lst = []  
    for i in range(n):  
        lst.append(i)  
    sum = 0  
    for elem in lst:  
        sum = sum + crash(n) + crash(n)  
    return sum
```

Summer 2012 Final | Q2(b) Solutions

```
def crash(n): # O(n) function
    if n < 1:
        return n
    return crash(n - 1) * n
```

```
def into_me(n):
    lst = []
    for i in range(n): # O(n)
        lst.append(i)
    sum = 0
    for elem in lst: # do n times:
        sum = sum + crash(n) + crash(n)
    return sum
```

What is the order of growth in n of the runtime of `into_me`, where n is its input?

Answer: $O(n^2)$. We make $2n$ `crash` calls per `into_me` call, and the growth function of `crash` is $O(n)$.

Spring 2014 Final | Q5(c)

Give worst-case asymptotic $\Theta(\cdot)$ bounds – **you guys can write them as $O(\cdot)$ bounds** – for the running time of the following code snippets. As a reminder, it is meaningful to write things with multiple arguments like $\Theta(a + b)$, which you can think of as “ $\Theta(N)$ where $N = a + b$.”

```
def a(m, n):  
    for i in range(m):  
        for j in range(n // 100):  
            print("hi")
```

```
def b(m, n):  
    for i in range(m // 3):  
        print("hi")  
    for j in range(n * 5):  
        print("bye")
```

```
def d(m, n):  
    for i in range(m):  
        j = 0  
        while j < i: j = j + 100
```

```
def f(m):  
    i = 1  
    while i < m:  
        i = i * 2  
    return i
```

Spring 2014 Final | Q5(c) Solutions

Give worst-case asymptotic $\Theta(\cdot)$ bounds – **you guys can write them as $O(\cdot)$ bounds** – for the running time of the following code snippets. As a reminder, it is meaningful to write things with multiple arguments like $\Theta(a + b)$, which you can think of as “ $\Theta(N)$ where $N = a + b$.”

```
def a(m, n): # Answer:  $O(mn)$ .
    for i in range(m):
        for j in range(n // 100):
            print("hi")
```

```
def b(m, n): # Answer:  $O(m + n)$ .
    for i in range(m // 3):
        print("hi")
    for j in range(n * 5):
        print("bye")
```

```
def d(m, n): # Answer:  $O(m^2)$ .
    for i in range(m): # essentially  $1 + \dots + m$  work
        j = 0
        while j < i: j = j + 100
```

```
def f(m): # Answer:  $O(\log m)$ .
    i = 1
    while i < m:
        i = i * 2
    return i
```

Thanks, everyone!
Good luck on the final.

Recommended Reading

— — —

Unfortunately a late addition to these slides – but the 61A wiki has a pretty nice [writeup](#) on orders of growth.