EE 126 — Probability and Random Processes

March 08, 2018 — Poisson Process II

1 Lecture

Recap

A forward Markov chain says "given today, tomorrow doesn't depend on yesterday." A backward Markov chain says "given today, yesterday doesn't depend on tomorrow." This is the same statement (see HW6), so a Markov chain in reverse is always a Markov chain. But it is not necessarily *reversible*.

Poisson Process

In a Poisson process, the arrival time $T_n = S_1 + S_2 + ... + S_n$ (the sum of the inter-arrival times, which are i.i.d. exponential).

Recall: Exp(λ) is memoryless. We have $P(\tau \le t) = 1 - e^{-\lambda t}$, $P(\tau > t) = e^{-\lambda t}$, and $P(\tau > t+s \mid \tau > t) = P(\tau > s)$. We also have $P(\tau \le t+\epsilon \mid \tau > t)$ ["success happened between t and ϵ"] $= \lambda\epsilon + o(\epsilon)$. $o(\epsilon)$ is defined as something that goes to 0 more quickly than ϵ, i.e. $\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0$.

Then

$$\begin{aligned} &P(\tau > t+\epsilon \mid \tau > t) = P(\tau > \epsilon) = e^{-\lambda\epsilon} = 1 - \lambda\epsilon + o(\epsilon) \\ &P(\tau < t+\epsilon \mid \tau > t) = \lambda\epsilon + o(\epsilon) \end{aligned}$$

$$\boxed{\begin{aligned} P(\text{no arrival in } (t, t+\epsilon)) &= 1 - \lambda\epsilon + o(\epsilon) \\ P(\text{1 arrival in } (t, t+\epsilon)) &= \lambda\epsilon + o(\epsilon) \\ P(\ge 2 \text{ arrivals in } (t, t+\epsilon)) &= o(\epsilon) \end{aligned}}$$

Theorem. The Poisson process is memoryless. It inherits this property from the exponential which defines it.

Let $N = \{N_t, t > 0\}$ be a Poisson process with parameter λ. Fix $t > 0$. Given $\{N_s, s \le t\}$ for $s \ge 0$, the process $\{N_{s+t} - N_t\}$ for $s \ge 0$ is also a Poisson process with parameter λ.

A key implication of this is that PP(λ) has *independent* and *stationary* increments. ("Independent" means that two disjoint intervals are independent. "Stationary" means that if we slide in time, the intervals don't change.) For any $0 \le t_1 < t_2 < ...$, $\{N_{t_{n+1}} - N_{t_n}\}$ are independent and the distribution depends only on the interval length $t_{n+1} - t_n$.

Theorem. If $N = \{N_t, t \ge 0\}$ is a PP(λ) random process, then N_t has a Poisson(λt) distribution.

Proof. We will make use of the density of jumps. Let T_n be the nth jump, and let $S_n = T_n - T_{n-1}$ be the inter-arrival time between $(n-1)$ and n. Let's calculate the joint density of the arrivals:

$$\begin{aligned} &P(T_1 \in (t_1, t_1+dt_1), T_2 \in (t_2, t_2+dt_2), ..., T_n \in (t_n, t_n+dt_n)) \\ &= P(S_1 \in (t_1, t_1+dt_1), S_2 \in (t_2-t_1, t_2-t_1+dt_2), ..., S_n \in (t_n - t_{n-1}, t_n - t_{n-1} + dt_n), S_{n+1} > t - t_n) \\ &= (\lambda e^{-\lambda t_1} dt_1)(\lambda e^{-\lambda(t_2-t_1)} dt_2) \cdots (\lambda e^{-\lambda(t_n - t_{n-1})} dt_n) \cdot e^{-\lambda(t-t_n)} \\ &= \lambda^n dt_1 dt_2 \cdots dt_n \cdot e^{-\lambda t} \end{aligned}$$

Therefore $P(N_t = n) = e^{-\lambda t}\lambda^n \int_S dt_1 dt_2 \cdots dt_n$. Note that n jumps in $(0,t)$ are equally likely in $(0,t)$, and S is the volume of $\{t_1, t_2, ..., t_n \mid 0 < t_1 < t_2 < ... < t_n < t\} \subset [0,t]^n$ corresponding to a specific ordering of $t_1, t_2, ..., t_n$. There are $n!$ such orderings, which are equally likely by symmetry. So $S = {}^{\text{Volume}(0,t)^n}/_{n!} = {}^{t^n}/_{n!}$, and

$$P(N_t = n) = e^{-\lambda t}\lambda^n \int_S dt_1 dt_2 \cdots dt_n = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Example: fish. Bob catches fish according to a Poisson process of rate $\lambda = 0.6$ fish/hr. If he catches at least one fish in the first two hours, he quits. Otherwise, he continues until he has caught his first fish.

What is the probability that Bob catches at least two fish? Since this can only happen if he catches two fish in the first two hours,

$$\begin{aligned} P(\text{Bob catches at least two fish}) &= 1 - P(0 \text{ fish in } (0,2)) - P(1 \text{ fish in } (0,2)) \\ &= 1 - e^{-1.2} - 1.2e^{-1.2} \end{aligned}$$

What is the expected number of fish that Bob catches? "Total number of fish caught" is equal to the number of fish caught in $(0,2)$ plus the number of fish caught in $(2,\infty)$. Therefore

$$\begin{aligned} \mathbb{E}[\text{fish caught by Bob}] &= \mathbb{E}[\#\text{ fish in } (0,2)] + \mathbb{E}[\#\text{ fish in } (2,\infty)] \\ &= 1.2 + (1)(e^{-1.2}) \end{aligned}$$

Splitting and Merging of Poisson Process

Suppose we have a Poisson process with rate λ which is split according to probabilities p and $1-p$. Then each of the splits will also be Poisson processes (with rates λp and $\lambda(1-p)$, respectively).

It's the same deal if we merge two Poisson processes (with rates λ_1 and λ_2) into one stream. The merged stream will be a Poisson process with rate $\lambda_1 + \lambda_2$.