# CSE 274: Radiometry and BRDFs

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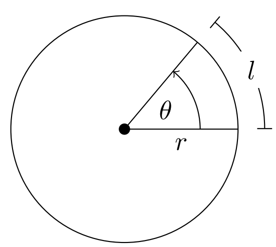
## 1 Radiometry

From a computer graphics perspective, we can think of **light** as visible electromagnetic radiation (EMR), i.e. EMR with wavelengths between (about) 400 and (about) 700 nm. Property-wise, we're mostly concerned with wavelengths and amount of energy transmitted; there are many other properties of light such as polarization, quantum effects, and wave behavior which we usually ignore.

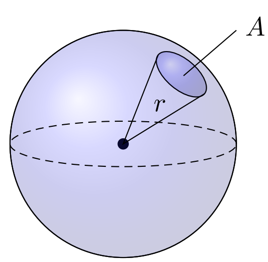
**Radiometry** is a system for measuring light as physical quantities such as radiance, irradiance, etc. It accounts for both spatial and angular properties of light, i.e. quantities are often parameterized or defined in terms of positions and directions.

### 1.1 Solid Angles

To discuss the angular distribution of light, we must first define the concept of **solid angles**. Let's start with angles, of which solid angles are the 3D analogue.




Here, the angle $\theta$ is $l/r$ radians. On a unit sphere, it is a cut of the circumference.




Here, the solid angle $\omega$ is $A/r^2$ steradians ($sr$). On a unit sphere, it is a cut of the surface area, meaning a solid angle is equivalent to projected area on the surface of a unit sphere. It might represent a bundle of rays coming out of the center of the sphere around a direction. With a total surface area of $4\pi r^2$, a sphere contains $4\pi$ $sr$.

We can define a direction on a sphere in terms of spherical coordinates $(\theta, \phi)$, where $\theta$ is the elevation (angle down from the north pole) and $\phi$ is the azimuth (angle to the left of an arbitrary point). Depending on you define the $x$, $y$, and $z$-axes, a point $(x, y, z)$ on the unit sphere can be defined as $(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ (meaning, e.g., $z = 0$ on the equator where $\theta = \pi/2$).

Note: if $\theta$ is the angle between a surface normal and the direction of a light source, we'll likely have to apply a $\cos\theta$ term as well, something like $d\omega\cos\theta$. (We get the most light when it's falling perpendicularly on the surface, i.e. $\theta = 0$, and less and less as the angle deviates from this.)

With integration in mind, we might also want to define the concept of a **differential solid angle** $d\omega = dA/r^2$, where $dA$, a rectangular differential area on a sphere with radius $r$, obtained by varying $\theta$ by $d\theta$ and $\phi$ by $d\phi$, is $(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,d\theta\,d\phi$. And then $d\omega = \sin\theta\,d\theta\,d\phi$.

*Side note*: this is equivalent to $-dz\,d\phi$, as shown –

$$\begin{aligned}(\sin\theta\,d\theta)\,d\phi &= -d(\cos\theta)\,d\phi \\ &= -dz\,d\phi\end{aligned}$$

With $d\omega$, we can perform integration on spheres.
For example, we can confirm that the total solid angle of a sphere $S^2$ is $4\pi$:

$$\begin{aligned}\int_{S^2} d\omega &= \int_{\phi=0}^{2\pi}\left(\int_{\theta=0}^{\pi}\sin\theta\,d\theta\right)\,d\phi \\ &= 2\pi\int_0^{\pi}\sin\theta\,d\theta \\ &= -2\pi\cos\theta\,\Big|_0^{\pi} \\ &= 4\pi\end{aligned}$$

## 1.2 Radiance

**Radiance** is the physical quantity associated with light along a ray. It is power per unit (projected) area flowing through a surface at $\mathbf{x}$, per unit solid angle in the direction $\omega$. Its units are $\frac{W}{m^2\cdot sr}$.

It is denoted $L(\mathbf{x}, \omega)$, which by no coincidence is the same as the 5D plenoptic function (note: $\mathbf{x}$ is 3D, $\omega$ is 2D). The plenoptic function, i.e. the light field, describes all of the radiance in the environment (amount of light flowing in every direction through every point), and in fact every other radiometric quantity can be derived from radiance. If you know the radiance distribution in the scene, you can get the irradiance, the radiant exitance, etc.

Notice that we refer to "projected" area. The surface at $\mathbf{x}$ (if there is one) might be oriented any which way, but radiance is measured perpendicularly to the $\omega$ direction, so we only care about flux flowing through the perpendicular component of the (hypothetical) surface. If the actual passed-through area is $dA$, then the perpendicular component is $dA\ \cos\theta$.

$$\begin{aligned}L(\mathbf{x}, \omega) &= \frac{d^2\Phi}{d\omega\,dA\ \cos\theta} \\ d^2\Phi &= L(\mathbf{x}, \omega)\,d\omega\,dA\ \cos\theta \quad \text{“directional power arriving at surface”}\end{aligned}$$

The total amount of flux (power) through a surface can be determined by integrating over the entire surface area and over all solid angles of light:

$$\Phi = \int_A\int_\Omega L(\mathbf{x}, \omega)\,d\omega\,dA\ \cos\theta$$

### 1.2.1 Radiance Properties

- In free space, radiance remains constant along lines [i.e. as it travels along a ray].
- Sensor response is proportional to radiance; a pixel intensity corresponds to the radiance of light at that pixel (times a throughput scaling factor).

## 1.3 Irradiance

Radiance $L = \frac{d^2\Phi}{d\omega\, dA\, \cos\theta}$ is flux per unit area per unit solid angle. **Irradiance** $E = \frac{d\Phi}{dA}$ is flux per unit area, and represents all light falling on a surface at a point. Its units are $\frac{W}{m^2}$. Note that radiance is the derivative of irradiance; namely it is irradiance per unit solid angle.

$$\begin{aligned} L_i(\mathbf{x}, \omega) &= \frac{dE(\mathbf{x}, \omega)}{d\omega\ \cos\theta} \\ dE(\mathbf{x}, \omega) &= L_i(\mathbf{x}, \omega)\, d\omega\ \cos\theta \end{aligned}$$

$dE$ is the power per unit area given by direction $\omega$. The $i$ means "incident." To obtain irradiance $E$, we can integrate this over all directions.

For example, total irradiance (integrated over a hemisphere $H^2$) is

$$\begin{aligned} E &= \int L_i(\mathbf{x}, \omega)\ \cos\theta\, d\omega \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} L_i(\mathbf{x}, \theta, \phi)\ \cos\theta \sin\theta\, d\theta\, d\phi \end{aligned}$$

Note: we can photograph a mirror sphere and obtain the distribution of incident radiance $L_i(\mathbf{x}, \theta, \phi)$ for any $\theta, \phi$ as an **illumination environment map**. This allows us to quickly compute the irradiance integral above *for a surface oriented in any direction* (giving an **irradiance environment map**). The former map is particularly useful for reflective surfaces; the latter for diffuse surfaces. *This method assumes faraway lighting; the ball's position in the scene shouldn't affect the radiances.*

If $L_i(\mathbf{x}, \theta, \phi)$ is constant, i.e. the same amount of light is coming in from each direction, then

$$\begin{aligned} E &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} L\ \cos\theta \sin\theta\, d\theta\, d\phi \\ &= 2\pi L \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta\, d\theta \\ &= 2\pi L \int_0^1 u\, du \quad (\text{for } u = \sin\theta, du = \cos\theta\, d\theta) \\ &= 2\pi L \left(\frac{1}{2}\right) \\ &= \pi L\ \left[W/m^2\right] \end{aligned}$$

If we have a **uniform** incoming radiance of 1 $\frac{W}{m^2 \cdot sr}$, the resulting irradiance is $\pi\ \frac{W}{m^2}$. Now, for a **diffuse** surface, reflected light (in every direction) is proportional to irradiance i.e. as $\rho E$.

$\rho$ is a constant Lambertian coefficient, technically the surface's BRDF, which controls how much light is reflected in any one direction. For energy conservation, i.e. to reflect at most 1 $\frac{W}{m^2 \cdot sr}$ back out, the maximum Lambertian value possible is $\rho_{\max} = \frac{1}{\pi}$.

### 1.4 Radiant Exitance

Radiant exitance, also known as **radiosity**, is the power per unit area leaving the surface. Its units are the same as irradiance (the only difference is that irradiance is for *light arriving*, and radiant exitance is for *light leaving*).

## 2 BRDFs

Different surfaces reflect light in different ways. As examples,

- **ideal specular (mirror)** reflects in mirror direction
- **ideal diffuse (matte)** reflects equally in all directions
- **specular (glossy)** reflects in lobe generally around mirror direction

The **BRDF**, or bidirectional reflectance distribution function, gives us the fraction of light from an incident direction $\omega_i$ which is reflected in an outgoing direction $\omega_r$. It is denoted $f(\omega_i, \omega_r)$.

Concretely, the BRDF is a *material-specific* ratio between outgoing radiance and incoming irradiance.

$$\begin{aligned} dL_r(\omega_r) &= f(\omega_i, \omega_r) dE_i(\omega_i) \\ dL_r(\omega_r) &= f(\omega_i, \omega_r) dL_i(\omega_i) \cos\theta_i \, d\omega_i \\ f(\omega_i, \omega_r) &= \frac{dL_r(\omega_r)}{dE_i(\omega_i)} \end{aligned}$$

Since $\omega_i$ and $\omega_r$ are parameterized by two angles each, the BRDF is a function of four variables.

$$f(\theta_i, \phi_i, \theta_r, \phi_r) \quad \textit{you might see it written like this}$$

### 2.1 BRDF Properties

- **Linearity.** If you have a material with a diffuse component and a specular component, you can obtain the net BRDF by summing the individual BRDFs.
- **Helmholtz reciprocity.** You can swap incoming and outgoing directions: $f(\omega_i \to \omega_r) = f(\omega_r \to \omega_i)$. Note: while the *BRDF* is symmetric, the net lighting effect typically isn't, e.g. it won't make $dL_r$ symmetric because $dL_r$ includes the cosine term.
- **Isotropy.** Many BRDFs are isotropic (can rotate surface around the normal without changing appearance). Then what matters is not the actual values of $\phi_i$ and $\phi_r$, but the relative difference. In other words, $f(\theta_i, \phi_i, \theta_r, \phi_r) = f(\theta_i, \theta_r, \phi_r - \phi_i)$.
- **Energy conservation.** The radiant exitance should not exceed the incoming irradiance.

### 2.2 Reflection Equation

The reflection equation tells us the reflected light as a function of the incident light. (Basically, we just integrate $dL_r$ to get $L_r$.)

$$\underbrace{L_r(\mathbf{x}, \omega_r)}_{\text{reflected light}} = \underbrace{L_e(\mathbf{x}, \omega_r)}_{\text{emission}} + \int_\Omega \underbrace{L_i(\mathbf{x}, \omega_i)}_{\text{incident light}} \underbrace{f(\mathbf{x}, \omega_i, \omega_r)}_{\text{BRDF}} \cos\theta_i \, d\omega_i$$

**Emission term.** Some surfaces in the scene should emit light.

## 2.3 BRDF Models

### 2.3.1 Ideal Diffuse Reflectance

- The reflected light in one direction is $fE$.
- Total reflected light in all directions is $\pi fE$, or $\pi fE\ \cos\theta$ by Lambert's cosine law.
- Let the **albedo** (ratio of reflected radiance to incident irradiance) be $\rho = \frac{\pi fE}{E} = \pi f$.
  - Total reflected light becomes $\rho E\ \cos\theta$.
  - The BRDF becomes $\frac{\rho}{\pi}$, i.e. the diffuse albedo $\rho \in [0, 1]$ divided by $\pi$.

### 2.3.2 Fresnel Reflectance

In the Fresnel reflectance model, the specular reflection of a surface depends on the viewing direction. In the example of the book on the table, the reflectivity increases (i.e. it appears to reflect more of the world) as you look at the surface from an increasingly grazing angle.

### 2.3.3 Torrance-Sparrow Reflectance

A compound specular reflectance model which includes a Fresnel term $F$, a geometric attenuation factor $G$, a distribution term $D$, and cosine terms which account for foreshortening.

$$f = \frac{FGD}{4\cos\theta_i\ \cos\theta_r}$$

Actually, the above BRDF is the general form for microfacet reflection models (I haven't fully specified the $F$, $G$, or $D$ terms e.g. for Torrance-Sparrow).

### 2.3.4 Empirical

We can define an empirical BRDF model by building a 4D table using measurements from a gantry.