CSE 252B Lecture 03

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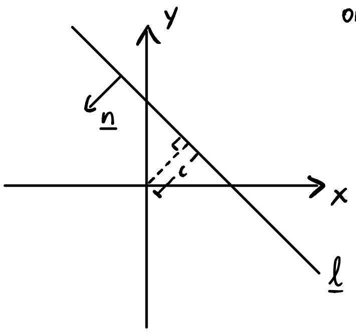
2D LINES

inhomogeneous line equation: $a\tilde{x} + b\tilde{y} + c = 0$

homogeneous line equation: $a\frac{x}{w} + b\frac{y}{w} + c = 0 \rightarrow ax + by + cw = 0$



or: $\underline{x}^T \underline{l} = 0$ (with $\underline{l}$ a homogeneous entity)

where

$$\underline{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}, \quad \underline{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \underline{n} \\ c \end{bmatrix}$$

Note: distance from origin is $\frac{c}{\|\underline{n}\|}$

Line at infinity: $\underline{l}_\infty = (0, 0, 1)^T$

3D PLANES

plane equation: $aX + bY + cZ + dT = 0$

$\rightarrow \underline{\pi}^T \underline{X} = 0$ where $\underline{\pi} = (a, b, c, d)^T = (\underline{n}^T, d)^T$

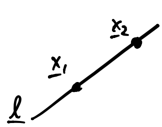
Plane at infinity: $\underline{\pi}_\infty = (0, 0, 0, 1)^T$

2D POINTS AND LINES

$\begin{cases} \underline{x}^T \underline{l} = 0 \\ \underline{l}^T \underline{x} = 0 \end{cases}$ iff $\underline{x}$ is on the line $\underline{l}$

e.g. $\underline{x}_1^T \underline{l} = 0$

$\underline{x}_2^T \underline{l} = 0$



$\begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \end{bmatrix} \underline{l} = \underline{0}$; $\underline{l}$ is right null space of $\begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \end{bmatrix}$

$\begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \end{bmatrix}$ is left null space of $\underline{l}$

JOIN of points

$\underline{l} = \underline{x}_1 \times \underline{x}_2$ (only valid b/c 3-vectors)

$= [\underline{x}_1]_\times \underline{x}_2$ (cross product is null space / orthogonal vector to $\underline{x}_1$ and $\underline{x}_2$)

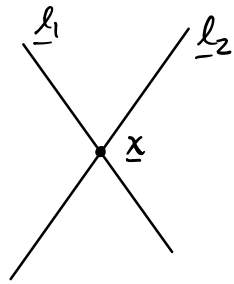
$= (\underline{x}_1^T [\underline{x}_2]_\times)^T$

$[\underline{l}]_\times = \underline{x}_1 \underline{x}_2^T - \underline{x}_2 \underline{x}_1^T$; for $\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $[\underline{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

skew-symmetric cross product matrix

INTERSECTION of lines

Given two lines, find the $\underline{x}$ s.t. $\begin{bmatrix} \underline{\ell}_1^T \\ \underline{\ell}_2^T \end{bmatrix} \underline{x} = \underline{0}$



$$\underline{x} = \underline{\ell}_1 \times \underline{\ell}_2$$
$$= [\underline{\ell}_1]_\times \underline{\ell}_2$$
$$= (\underline{\ell}_1^T [\underline{\ell}_2]_\times)^T$$

$$[\underline{x}]_\times = \underline{\ell}_1 \underline{\ell}_2^T - \underline{\ell}_2 \underline{\ell}_1^T$$

3D POINTS AND PLANES

$$\begin{cases} \underline{X}^T \underline{\pi} = 0 \\ \underline{\pi}^T \underline{X} = 0 \end{cases}$$ iff $\underline{X}$ is on the 3D plane $\underline{\pi}$

Points on a plane

$$\underline{X}_1^T \underline{\pi} = 0$$
$$\underline{X}_2^T \underline{\pi} = 0 \quad \rightarrow \quad \underset{3\times4}{\begin{bmatrix} \underline{X}_1^T \\ \underline{X}_2^T \\ \underline{X}_3^T \end{bmatrix}} \underset{4\times1}{\underline{\pi}} = \underset{3\times1}{\underline{0}}$$
$$\underline{X}_3^T \underline{\pi} = 0$$

so left null space of plane is 3 points that lie on plane

Planes intersecting at a point

$$\underline{\pi}_1^T \underline{X} = 0$$
$$\underline{\pi}_2^T \underline{X} = 0$$
$$\underline{\pi}_3^T \underline{X} = 0$$

planes that intersect at a point

$$\begin{bmatrix} \underline{\pi}_1^T \\ \underline{\pi}_2^T \\ \underline{\pi}_3^T \end{bmatrix} \underline{X} = 0$$

2D CONIC SECTIONS

general equation : $a\tilde{x}^2 + b\tilde{x}\tilde{y} + c\tilde{y}^2 + d\tilde{x} + e\tilde{y} + f = 0$ (inhomog. coords)
$ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0$ (homog. coords)

for a point $(x, y, w)^T$ on the conic,

$$[x \ \ y \ \ w] \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

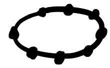
$$\underline{x}^T \underline{C} \underline{x} = 0$$

"point conics"
point must lie on conic for this to be true!

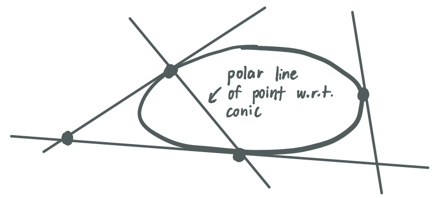
$\underline{C}$ is 5 DOF
homogeneous entity (defined up to scale)

CONICS AND DUAL CONICS

$\mathbf{x}^T C \mathbf{x} = 0$ gives points on C



- **pole**: point
- **polar line**: line which relates to pole according to the conic
  - if the point is on the conic, the polar will be a line tangent to that point
  - if the point is inside the conic, the polar will be a line outside of the conic
  - if the point is outside the conic, draw two lines and connect the tangent points to find the polar



$\mathbf{l} = C\mathbf{x}$: polar line of $\mathbf{x}$ w.r.t. C

provides fast way to get tangent line of conic:
just pick point on conic and you'll get a tangent line $\mathbf{l}$ from this

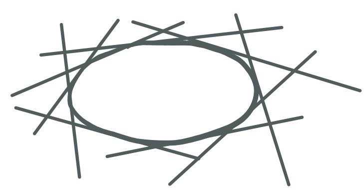
DUAL CONICS

just as points and hyperplanes are geometric duals, conics and dual conics are duals

dual conics are "line conics"

$\mathbf{l}^T C^* \mathbf{l} = 0$ where C^* describes a dual conic

- lines are tangent to C
- the conic is defined by an infinite # of lines tangent to it, which "carve it out" in space



$\mathbf{x} = C^* \mathbf{l}$: pole of $\mathbf{l}$ w.r.t. C

note that C^* is a matrix of cofactors of C;

$C^* = C^{-1}$ (up to scale) , $C = (C^*)^{-1}$ (up to scale)

The center of the conic is the pole of the line at infinity $\mathbf{l}_\infty = (0, 0, 1)^T$:

$\mathbf{x}_{center} = C^* \mathbf{l}_\infty$

meaning there is an easy way to find the center of an ellipse/conic:

just invert C and multiply by $\mathbf{l}_\infty$!

COFACTOR AND ADJOINT MATRICES

$M^* = \det(M) M^{-T}$ is a matrix of cofactors of M
$\quad = \mathrm{adj}(M)^T$

↖ adjugate (or classical adjoint) of a matrix

$(M^{-T} = (M^{-1})^T = (M^T)^{-1})$

so the "up to scale"s are because $M^* = \underset{\substack{\uparrow \\ \text{scale}}}{\det(M)} M^{-1}$

also $\det(\underline{M})\,\underline{M}^{-T} = \mathrm{adj}(\underline{M})^T$

$$\underline{M}^{-1} = \frac{1}{\det(\underline{M})}\,\mathrm{adj}(\underline{M})$$

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if $\underline{M}$ is symmetric (note that conics are symmetric),

$$\underline{M}^* = \det(\underline{M})\,\underline{M}^{-T}$$
$$= \det(\underline{M})\,\underline{M}^{-1}$$
$$= \mathrm{adj}(\underline{M})$$; compute this if $\underline{M}$ is singular

QUADRICS AND DUAL QUADRICS

quadric : quadratic surface
(scaling conics up in dimension)

$$\underline{X}^T \underline{Q}\,\underline{X} = 0$$

(analogue of $\underline{x}^T \underline{C}\,\underline{x} = 0$)

where $\underset{4\times1}{\underline{X}}$ is a point on the quadratic surface $\underset{4\times4}{\underline{Q}}$

$\underline{\pi} = \underline{Q}\underline{X}$: polar plane of $\underline{X}$ w.r.t. quadric $\underline{Q}$

if $\underline{X}$ is on $\underline{Q}$ surface,
$\underline{\pi}$ is tangent plane to that point

now, instead of lines carving out a conic section,
we have a bunch of (hyper)planes carving out a quadric

$\underline{\pi}^T \underline{Q}^* \underline{\pi} = 0$ planes tangent to $\underline{Q}$

$\underline{X} = \underline{Q}^* \underline{\pi}$ pole of $\underline{\pi}$ w.r.t. $\underline{Q}$

the center of the quadric is the pole of plane @
infinity $\underline{\pi}_\infty = (0, 0, 0, 1)^T$

LINES IN 3D

Plücker line coordinates

Plücker matrix

$$\underline{L} = \begin{bmatrix} 0 & l_{12} & l_{13} & l_{14} \\ -l_{12} & 0 & l_{23} & -l_{42} \\ -l_{13} & -l_{23} & 0 & l_{34} \\ -l_{14} & l_{42} & -l_{34} & 0 \end{bmatrix}$$

homage to original paper

(skew-symmetric matrix)

if you have $\underline{L}$, there are corresponding line coords

$$\underline{L} = (l_{12}, l_{13}, l_{14}, l_{23}, l_{42}, l_{34})$$

~~6 DOF~~ 6 unique entries
~~5 DOF~~ only defined up to scale
4 DOF $\det(\underline{L}) = 0$

3D LINES

can take dual of Plücker line matrix

$\underline{L}^* \underline{X} = 0$, $\underline{X}^T \underline{L}^{*T} = \underline{0}^T$ iff point $\underline{X}$ is on line $\underline{L}$

$\underline{L}\underline{\pi} = 0$, $\underline{\pi}^T \underline{L}^T = \underline{0}^T$ iff line $\underline{L}$ is on plane $\underline{\pi}$

Two planes intersect at a line

$$\underset{4\times4}{\underline{L}} \; \underset{4\times2}{[\underline{\pi}_1 \mid \underline{\pi}_2]} = \underset{4\times2}{\underline{0}}$$

two columns of null space : two planes that intersect at line

Two points on line

$$\underline{L}^* [\underline{X}_2 \mid \underline{X}_1] = \underline{0}$$

Join of two points

$$\underline{L} = \underline{X}_1 \underline{X}_2^T - \underline{X}_2 \underline{X}_1^T$$

Intersection of line and plane

$$\underline{X} = \underset{4\times4}{\underline{L}} \; \underset{4\times1}{\underline{\pi}}$$

fast way to intersect line and plane; makes this all worth it (from a graphics perspective)

dual

$$\underline{L}^* = \begin{bmatrix} 0 & l_{34} & l_{42} & l_{23} \\ -l_{34} & 0 & l_{14} & -l_{13} \\ -l_{42} & -l_{14} & 0 & l_{12} \\ -l_{23} & l_{13} & -l_{12} & 0 \end{bmatrix}$$

direction $\underline{d}$ of line

$$\underline{X}_\infty = \underline{L}\,\underline{\pi}_\infty$$

$$(\underline{d}^T, 0)^T = \underline{L}(0,0,0,1)^T$$

LINE AS PENCIL OF POINTS

Given two distinct points $\underline{x}_1$ & $\underline{x}_2$ on a line,

(any dim, but here 2D)

$\lambda \underline{x}_1 + \mu \underline{x}_2$ is the pencil of points on the line

another way to parameterize a line which is defined by the join of points

Different ways to represent this (all valid)

$$\underline{x}(\lambda, \mu) = \lambda \underline{x}_1 + \mu \underline{x}_2$$

$$\underline{x}(\lambda) = \lambda \underline{x}_1 + (1-\lambda)\underline{x}_2$$

$\underline{x}(\lambda) = \underline{x}_1 + \lambda \underline{D}$ where $\underline{D} = (\underline{d}^T, 0)^T$ with direction $\underline{d}$

inhomog. coords: $\tilde{\underline{x}}(\lambda) = \tilde{\underline{x}}_1 + \lambda \underline{d}$ where $\underline{d}$ is a direction vector

LINE - CONIC INTERSECTION

(line will intersect at 0, 1, or 2 points)

1. Determine 2 points $\underline{x}_1$ and $\underline{x}_2$ on the line $\underline{\ell}$.

$$\underline{\ell}^T [\underline{x}_1 \mid \underline{x}_2] = \underline{0}^T \qquad \text{or} \qquad \begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \end{bmatrix} \underline{\ell} = \underline{0}$$

$$\underline{\ell}^T \underline{M} = \underline{0}^T \qquad \qquad \underline{M}^T \underline{\ell} = \underline{0}$$

2. Plug into "pencil of points" representation.

$$\underline{x}(\lambda) = \lambda \underline{x}_1 + (1-\lambda)\underline{x}_2$$

3. Plug into conic equation.

$$\underline{x}^T \underline{C} \underline{x} = 0$$
$$\underline{x}(\lambda)^T \underline{C} \underline{x}(\lambda) = 0$$

4. Solve for λ.

There will be 0, 1, or 2 solutions, which represent the points of intersection between the line and the conic.

Note that our $\underline{x}(\lambda)$ equation gives us a point for each value of λ we put in.

$$(\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2)^T \underline{C} (\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2) = 0$$
$$(\underline{x}_1^T \underline{C} \underline{x}_1 - 2\underline{x}_1^T \underline{C} \underline{x}_2 + \underline{x}_2^T \underline{C} \underline{x}_2)\lambda^2 + [2(\underline{x}_1^T \underline{C} \underline{x}_2 - \underline{x}_2^T \underline{C} \underline{x}_2)]\lambda + \underline{x}_2^T \underline{C} \underline{x}_2 = 0$$

2 solutions λ_1, λ_2, of which 0, 1, or 2 are real

5. Substitute the λs back into the $\underline{x}(\lambda)$ equation.

This will give you the intersection points on the conic.

$$\left.\begin{aligned} \underline{x}_{c_1} &= \lambda_1 \underline{x}_1 + (1-\lambda_1)\underline{x}_2 \\ \underline{x}_{c_2} &= \lambda_2 \underline{x}_1 + (1-\lambda_2)\underline{x}_2 \end{aligned}\right\} \text{2 intersection points}$$