1 Course Overview

- The focus of the course is 3D computer graphics – to understand the theory of it, and to be able to write 3D graphics programs in OpenGL, etc.
- Reasons to study computer graphics: (1) intellectual challenge, (2) lots of applications.
  - *Applications*: movies, games, CAD (e.g. airplanes), interior design, medical visualization, VR...

2 3D Graphics Pipeline

- There are three stages in the graphics pipeline: (1) modeling, (2) animation, and (3) rendering.
  - *Modeling*: creating a geometric model of an object, e.g. a polygon mesh exported from Blender.
  - *Animation*: consideration of the motion of the model(s).
  - *Rendering*: turning the models and animation into realistic images, e.g. modeling light transport.
- A note as part of rendering: (1) rasterization and (2) ray tracing are two ways to create images.
  - *Rasterization*: take each geometric primitive, figure out where it goes on the screen.
  - *Ray tracing*: take each point on the screen, figure out which geometric primitive it corresponds to.
  - Thus rasterization and ray tracing do things in opposite directions.
  - Typically, ray tracing produces higher-quality images but is slower.

3 Basic Math

- Vectors have lengths and directions, and are used to represent offsets.
- Vectors are used to represent positions as well, although positions should technically not be vectors because vectors are independent of the origin and do not have any notion of absolute position.
- One common use of the dot product is to compute the angle between two vectors. It's easy to calculate a dot product, and by extension an L2-norm (which is just the square root of a vector dotted with itself). Thus, we are able to calculate the angle ϕ between vectors $\mathbf{a}$ and $\mathbf{b}$ as

$$\phi = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right)$$

- Another application of the dot product: projecting one vector onto another, e.g. when determining the coordinates of a point in a new coordinate system. To project $\mathbf{b}$ onto $\mathbf{a}$, we can compute

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\|\cos\phi\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}\right) = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}$$

- If we have two vectors $\mathbf{a}$ and $\mathbf{b}$, then the cross product $\mathbf{a}\times\mathbf{b}$ will be orthogonal to $\mathbf{a}$ and $\mathbf{b}$. Its length will be given by the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$, i.e. $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{b}\|\sin\phi\cdot\|\mathbf{a}\|$.

- The direction of $\mathbf{a} \times \mathbf{b}$ is given by the right-hand rule: curve your hand from $\mathbf{a}$ to $\mathbf{b}$, and the direction of your thumb will be the direction of $\mathbf{a} \times \mathbf{b}$. Accordingly, order matters! And $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
  - Of note: you have to curve the short way around (through ϕ), not the long way.
- A few cross product properties:
  - If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ represent the unit axes in an orthogonal RH coordinate system,

    $$\begin{array}{lll} \mathbf{x} \times \mathbf{y} = \mathbf{z} & \mathbf{y} \times \mathbf{z} = \mathbf{x} & \mathbf{z} \times \mathbf{x} = \mathbf{y} \\ \mathbf{y} \times \mathbf{x} = -\mathbf{z} & \mathbf{z} \times \mathbf{y} = -\mathbf{x} & \mathbf{x} \times \mathbf{z} = -\mathbf{y} \end{array}$$

    (To remember the positive case, go in a cyclical order from $\mathbf{x}$ to $\mathbf{y}$ to $\mathbf{z}$.)
  - Distributivity over addition: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
  - $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ because $\sin\phi$ (and therefore the length of the vector) is equal to 0
  - $\mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$
- Here's a derivation for the (3D) Cartesian formula of the cross product:

  $$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (x_a\mathbf{x} + y_a\mathbf{y} + z_a\mathbf{z}) \times (x_b\mathbf{x} + y_b\mathbf{y} + z_b\mathbf{z}) \\ &= x_a y_b(\mathbf{x} \times \mathbf{y}) + x_a z_b(\mathbf{x} \times \mathbf{z}) + y_a x_b(\mathbf{y} \times \mathbf{x}) + y_a z_b(\mathbf{y} \times \mathbf{z}) + z_a x_b(\mathbf{z} \times \mathbf{x}) + z_a y_b(\mathbf{z} \times \mathbf{y}) \\ &= x_a y_b(\mathbf{z}) + x_a z_b(-\mathbf{y}) + y_a x_b(-\mathbf{z}) + y_a z_b(\mathbf{x}) + z_a x_b(\mathbf{y}) + z_a y_b(-\mathbf{x}) \\ &= y_a z_b\mathbf{x} - z_a y_b\mathbf{x} + z_a x_b\mathbf{y} - x_a z_b\mathbf{y} + x_a y_b\mathbf{z} - y_a x_b\mathbf{z} \\ &= (y_a z_b - z_a y_b)\mathbf{x} + (z_a x_b - x_a z_b)\mathbf{y} + (x_a y_b - y_a x_b)\mathbf{z} \end{aligned}$$

  In other words,

  $$\begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \times \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} y_a z_b - z_a y_b \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{bmatrix}$$

  This can also be written as a matrix-vector product as

  $$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix}$$

4 Coordinate Systems

- In computer graphics, there are often many sets of coordinate systems, e.g. one for the world, one for a 3D model, one for parts of a 3D model (head, shoulders, etc.)...
- A coordinate frame is defined as a set of three 3D vectors $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ s.t.
  - the vectors are of unit length
  - the vectors are mutually orthogonal (and $\mathbf{u} \times \mathbf{v} = \mathbf{w}$)
- A vector $\mathbf{p}$ can be written in terms of its projections onto these axes:

  $$\mathbf{p} = (\mathbf{p} \cdot \mathbf{u})\mathbf{u} + (\mathbf{p} \cdot \mathbf{v})\mathbf{v} + (\mathbf{p} \cdot \mathbf{w})\mathbf{w}$$

- For example, a coordinate system might be $(\mathbf{w})$ an axis for the viewing direction, $(\mathbf{v})$ an axis for the up vector, and $(\mathbf{u})$ an axis orthogonal to $\mathbf{v}$ and $\mathbf{w}$.