Calculus	Stewart
Chapter 10	Parametric Equations & Polar Coordinates

1 Reading

10.1. Curves Defined by Parametric Equations

Traditionally, we write our functions as $y = f(x)$ or $x = f(y)$. But sometimes we might want to describe curves in terms of another parameter. We can represent a lot of weird curves (e.g. those which fail the vertical line test, like circles) more easily via equations parameterized by a new variable t. 2D curves can therefore be expressed as

$$(x, y) = (f(t), g(t))$$

i.e. as **parametric curves**. In general, this can be thought of a particle moving around the xy-plane as a function of time t.

10.2. Calculus with Parametric Curves

Tangents

Given a 2D curve $y = F(x)$ with parametric form $(x, y) = (f(t), g(t))$, we can find the slope of the tangent line at point (x, y) as

$$\begin{aligned} g(t) &= F(f(t)) \\ g'(t) &= F'(f(t))f'(t) \\ \frac{g'(t)}{f'(t)} &= F'(x) \\ \frac{dy/dt}{dx/dt} &= \frac{dy}{dx} \end{aligned}$$

Areas

The area under the curve $(x, y) = (f(t), g(t))$ over the domain $a \leq x \leq b$ (or $\alpha \leq t \leq \beta$) can be written

$$\int_a^b y\ dx = \int_\alpha^\beta g(t)f'(t)\ dt$$

after making the substitutions

$$\begin{aligned} y &= g(t) \\ x = f(t) \implies dx &= f'(t)dt \end{aligned}$$

Arc Length

The length L of a curve $(x, y) = (f(t), g(t))$ over the domain $a \leq x \leq b$ (or $\alpha \leq t \leq \beta$), for $dx/dt > 0$, is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{g'(t)}{f'(t)}\right)^2} f'(t)\ dt = \int_\alpha^\beta \sqrt{f'(t)^2 + g'(t)^2}\ dt$$

2 Exercises

References

[1] J. Stewart. *Calculus: Early Transcendentals*. Metric international version. Thomson Brooks/Cole, 2008.