Owen Jow **Spring 2018**

Fluid Simulation

An introduction to both classical and modern methods

1 Preface

These are my scattered notes on fluid simulation, which I am taking in preparation for transfer to a [hopefully more organized] book format. This document is not necessarily intended to be read by others, and therefore I do not plan to plot it out or inspect it very carefully. Do not expect all of the information to be cohesive.

On the other hand, for my own sake I will do my best to keep things presentable. :)

> It helps to have Sokal and Bricmont on hand to tell us the real reason why turbulent flow is a hard problem: the Navier-Stokes equations are difficult to solve.
>
> *Richard Dawkins*

2 Incompressible Navier-Stokes

The Navier-Stokes equations describe the motion of fluids. For instance, the momentum and incompressibility equations can be expressed as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p = \mathbf{g} + \nu \nabla \cdot \nabla \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0$$

where $\mathbf{u}$ is the fluid's velocity, ρ is the fluid's density, p is the pressure (force per unit area) that the fluid exerts on anything, $\mathbf{g}$ is the acceleration due to "forces," and ν is the viscosity.

2.1 The Momentum Equation

The first equation can be derived from Newton's second law of motion $\mathbf{F} = m\mathbf{a}$ for each particle. We treat the acceleration of a particle as $\frac{D\mathbf{u}}{Dt}$ (the material derivative). Then the forces acting on the particle are *gravity* ($m\mathbf{g}$), *pressure* ($-V\nabla p$) from the other particles, and *viscosity* ($V\mu\nabla \cdot \nabla \mathbf{u}$) from the other particles.

So $\mathbf{F} = m\mathbf{a}$ becomes

$$m\mathbf{g} - V\nabla p + V\mu\nabla \cdot \nabla \mathbf{u} = m\frac{D\mathbf{u}}{Dt}$$

After dividing by the volume V, dividing by the density $\rho = m/V$, and defining the kinematic viscosity ν as μ/ρ, this simplifies to

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p = \mathbf{g} + \nu\nabla \cdot \nabla \mathbf{u}$$

3 Lagrangian and Eulerian Specifications

In the Lagrangian frame of reference, a fluid is represented as a collection of particles, each with their own state (e.g. position and velocity). In the Eulerian frame of reference, a fluid is represented as a fixed-position grid over which the fluid flows.

It is much easier to approximate spatial derivatives on a Eulerian grid.

4 Marker-and-Cell Grid

A MAC grid, i.e. a *staggered* grid, samples pressure at the center of each cell and the velocity components at the edges.

5 Visualization

5.1 OpenGL

References

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"Fluid simulation," *ACM SIGGRAPH 2007 courses*, 1–81.

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